International Journal of Theoretical Physics, Vol. 44, No. 10, October 2005 (© 2005)
DOI: 10.1007/s10773-005-8894-1

Group Field Theory: An Overview[1]

L. Freidel[2,3]

We give a brief overview of the properties of a higher-dimensional generalization of matrix model which arise naturally in the context of a background approach to quantum gravity, the so-called group field theory. We show in which sense this theory provides a third quantization point-of-view on quantum gravity.

1. INTRODUCTION

Spin foam models describe the dynamics of loop quantum gravity in terms of state sum models. The purpose of these models is to construct the physical scalar product which is one of the main object of interest in quantum gravity. Namely, given a four-manifold M with boundaries Σ_0, Σ_1 and given a diffeomorphism class of three metric $[g_0]$ on Σ_0 and $[g_1]$ on Σ_1 we want to compute

$$\langle [g_0] | \mathcal{P} | [g_1] \rangle = \int_{\mathcal{M}} \mathcal{D}[g] e^{iS(g)}, \tag{1}$$

the integral being over $\mathcal{M}$: The space of all metrics on M modulo 4-diffeomorphism which agree with g_0, g_1 on ∂M. The action is the Einstein Hilbert action and $\mathcal{P}$ denotes the projector on the kernel of the hamiltonian constraint. This expression is of course highly formal, their is no good non perturbative[5] definition of the measure on $\mathcal{M}$ and no good handle on the space of kinematical states $|[g]\rangle$

In loop quantum gravity there is a good understanding of the kinematical Hilbert space (see Ashtekar and Lewandowski, 2004 for a review). In this framework the states are given by *spin networks* Γ_J where Γ is a graph embedded

[1] Prepared for the proceedings of Peyresq Physics 9 Meeting: Micro and Macro Structure of Spacetime, Peyresq, France, 19–26 June 2004.

[2] Perimeter Institute, 35 King Street North, Waterloo, Ontario, Canada N2J 2W9.

[3] Laboratoire de Physique, École Normale Supérieure de Lyon, 46 allée d'Italie, Lyon 69007, France.

[4] To whom correspondence should be addressed at Perimeter Institute, 35 King Street North, Waterloo, Ontario, Canada N2J 2W9; e-mail:lfreidel@perimeterinstitute.ca

[5] Except in 2 + 1 dimension (Witten, 1988).

in a three space Σ and $\jmath$ denotes a coloring of the edges of Γ by representations of a group G[6] and a coloring of the vertex of Γ by intertwiners (invariant tensor) of G. These states are eigenvectors of geometrical operators, the representations labeling edges of the spin network are interpreted as giving a quanta of area $l_p^2\sqrt{j(j+1)}$ to a surface intersecting Γ.

In this context the spacetime is obtained as a spin network history: If one evolve in time a spin network it will span a foam-like structure i.e., a combinatorial two-complex denoted $\mathcal{F}$. The edges of the spin network will evolve into faces of $\mathcal{F}$ the vertices of Γ will evolve into edges of $\mathcal{F}$ and transition between topologically different spin networks will occur at vertices of $\mathcal{F}$. The spin network coloring induces a coloring of $\mathcal{F}$: The faces of $\mathcal{F}$ are colored by representation $\jmath_{\rm f}$ of G and edges of $\mathcal{F}$ are colored by intertwiners $\iota_{\rm e}$ of G. Such a colored two complex $\mathcal{F}_{(\jmath_{\rm f},\iota_{\rm e})}$ is called a *spin foam* (Baez, 1998; Perez, 2003). By construction the boundary of a spin foam is an union of spin networks.

The definition is so far purely combinatorial, however if one restrict the two-dimensional complex $\mathcal{F}$ to be such that D faces meet at edges of $\mathcal{F}$ and $D+1$ edges meets at vertices of $\mathcal{F}$ we can reconstruct from $\mathcal{F}$ a D-dimensional piecewise-linear pseudo-manifold $M_{\mathcal{F}}$ with boundary (De Pietri, 2001; De Pietri and Petronio, 2000). Roughly speaking, each vertex of $\mathcal{F}$ can be viewed to be dual to a D-dimensional simplex and the structure of the two-dimensional complex gives the prescription for gluing these simplices together and constructing $M_{\mathcal{F}}$. The spin network states are dual to the boundary triangulation of $M_{\mathcal{F}}$.

A local *spin foam model* is characterized by a choice of local amplitudes $A_{\rm f}(\jmath_{\rm f})$, $A_{\rm e}(\jmath_{f_{\rm e}},\iota_{\rm e})$, $A_{\rm v}(\jmath_{f_{\rm v}},\iota_{e_{\rm v}})$ assigned respectively to the faces, edges and vertices of $\mathcal{F}$. $A_{\rm f}$ depends only on the representation coloring the face, $A_{\rm e}$ on the representations of the faces meeting at e and the intertwiner coloring the edge e, likewise $A_{\rm v}$ depends only on the representations and intertwiners of the faces and edges meeting at v.

Given a two complex $\mathcal{F}$ with boundaries Γ_0,Γ_1 colored by $\jmath_0,\jmath_1$ the Transition amplitude is given by

$$A(\mathcal{F}) = \langle \Gamma_{\jmath_0}|\Gamma_{\jmath_1}\rangle_{\mathcal{F}} \equiv \sum_{\jmath_{\rm f},\iota_{\rm e}} \prod_{\rm f} A_{\rm f}(\jmath_{\rm f}) \prod_{\rm e} A_{\rm e}(\jmath_{f_{\rm e}},\iota_{\rm e}) \prod_{\rm v} A_{\rm v}(\jmath_{f_{\rm v}},\iota_{e_{\rm v}}), \qquad (2)$$

the sum being over the labeling of internal faces and edges not meeting the boundary. Note that a priori the amplitude depend explicitly on the choice of the two complex $\mathcal{F}$.

There are many examples of such models. Historically, the first example is due to Ponzano and Regge (1968): They showed that the quantum amplitude for euclidean $2+1$ gravity with zero cosmological constant can be expressed as a spin foam model where the group G is SU(2), the faces are labeled by SU(2)

[6] In conventional loop quantum gravity the group is $SL(2,\mathbb{C})$, more generally G is a Lorentz group.

spin j_{f}[7] and the local amplitudes are given by $A_{\mathrm{f}}(j_{\mathrm{f}}) = (2j_{\mathrm{f}} + 1)$, $A_{\mathrm{e}}(j_{f_{\mathrm{e}}}) = 1$ and the vertex amplitude $A_{\mathrm{v}}(j_{f_{\mathrm{v}}})$ which depends on six spins is the normalized $6j$ symbol or Racah–Wigner coefficient. The remarkable feature of this model is that it does n't depend on the choice of the two complex $\mathcal{F}$ but only on $\mathcal{M}_{\mathcal{F}}$. The inclusion of cosmological constant or lorentzian gravity can be implemented easily by taking the group to be a quantum group (Turaev and Viro, 1992) or to be a noncompact Lorentz group (Freidel, 2000). Along the same line, it was shown that four-dimensional topological field theory called *BF* theory (Baez, 1996) can be quantized in terms of triangulation independent spin foam model (Ooguri, 1992).

It was first realized by Reisenberger that spin foam models give a natural arena to deal with four-dimensional quantum gravity (Reisenberger, 1994). Two seminal works triggered more interest on spin foam models: In the first one Barrett-Crane (1998) proposed a spin foam model for four-dimensional general relativity.[8] This model is obtained from the spin foam model of pure *BF* by restricting the Lorentz representation to be simple, so that the spin coloring the faces are SU(2) representations in the Euclidean context. In the second one, it was shown by Reisenberger and Rovelli (1997) that the evolution operator in loop quantum gravity can be expressed as a spin foam model and they propose an interpretation of the vertex amplitude in terms of the matrix elements of the hamiltonian constraint of loop quantum gravity (Thiemann, 1998). The spin labeling the faces are SU(2) representations and are interpreted as quanta of area.

It was soon realized that spin foam models can naturally incorporate causality (Markopoulou and Smolin, 1997), Lorentzian signature (Barrett and Crane, 2000) and coupling to gauge field theory (Oriti and Pfeiffer, 2002; Mikovic, 2002). The Barret–Crane prescription was understood to be linked to the Plebanski formulation of gravity where the Einstein action is written as a *BF* theory subject to constraints (De Pietri and Freidel, 1999). This formulation and the corresponding spin foam models was extended to gravity in any dimensions (Freidel *et al.*, 1999).

The main lesson is that spin foam is a very general structure which allows to address in a background independent manner the dynamical issues of a large class of diffeomorphism invariant models including gravity in any dimensions coupled to gauge fields (Freidel and Krasnov, 1999). This formulation naturally incorporated the fact that the kinematical Hilbert space of the theory is labeled by spin networks.[9]

[7] No edges intertwiner is needed since in three dimensions we restrict to only three face meeting along each edge and there is a unique normalized intertwiner between three SU(2) representation.

[8] More precisely a prescription for the vertex amplitude.

[9] This is relevant in view of the "LOST" uniqueness theorem stating that there is a unique diffeomorphism invariant representation (Sahlmann, 2002; Sahlmann and Thiemann, 2003; Okolow and Lewandowski, 2003) of a theory with phase space a pair of electric and magnetic field.

A different line-of-development originated from the detail study of the vertex amplitude proposed by Barret–Crane and the corresponding higher-dimensional quantum gravity models (Barrett, 1998; Freidel and Krasnov, 2000). These studies shows that these amplitudes can be written as some Feynman graph evaluation. For instance in the original Barrett-crane model

$$A_{\mathrm{v}}(j_1, \ldots, j_{10}) = \int_{S^3} \prod_{i=1}^{5} dx_i \prod_{i \neq j} G_{j_{ij}}(x_i, x_j), \tag{3}$$

where the 10 spins are simple representations of $SO(4)$ labeling the 10 faces of the four-simplex and $G_j(x, y)$ is the Hadamard propagator of S^3, $(\Delta_{S^3} + j(j+1))G_j = 0$, $G(x, x) = 1$.

This structure was calling a field theory interpretation of spin foam models. It was eventually found by De Pietri *et al.* (2000) that the Barrett–Crane spin foam model can remarkably be interpreted as a Feynman graph of a new type of theory baptized "group field theory" (GFT for short). The GFT structure was first discovered by Boulatov (1992) in the context of three-dimensional gravity where a similar connection was made and further developed by Ooguri in the context of 4d *BF* theory (Ooguri, 1992). Ambjorn, Durhuus, and Jonnson (1991) also pointed out similar structure in the context of dynamical triangulation. It is clear in this context that group field theory can be understood in a precise sense as a higher-dimensional generalization of matrix models which generates a summation over two-dimensional gravity models (Di Francesco *et al.*, 1995).

Reisenberger and Rovelli (2001) showed, in an key work, that the appearance of GFT in the context of spin foam models is not an accident but a generic feature. They proved that *any* local spin foam model of the form (2) can be interpreted as a Feynman graph of a group field theory.

We have argued that spin foam models generically appear in the context of background independent approach to quantum gravity (Freidel and Krasnov, 1999), this results shows that GFT is an important and unexpected universal structure behind the dynamics of such models. A deeper understanding of this theory is clearly needed. GFT was originally designed to address one of the main shortcomings of the spin foam approach: namely the fact that the spin foam amplitude (2) depends explicitly on the discrete structure (the two complex or triangulation). As we will now see in more details it does much more than that and give a third quantization point-of-view on gravity where spacetime is emergent and dynamical.

2. GROUP FIELD THEORY

2.1. Definition

In this section, we introduce the general GFT action that can be specialized to define the various spin foam models described in the introduction.

We consider a Lie group G which is the Lorentz group in dimension D ($G = \mathrm{SO(D)}$ for Euclidean gravity models and $G = \mathrm{SO}(D-1, 1)$ for Lorentzian ones.[10] D is the dimension of the spacetime and we will call the corresponding GFT a D-GFT. The *field* $\phi(x_1, \ldots, x_D)$, denoted $\phi(x_i)$ where $i = 1 \ldots D$, is a function on G^D. The dynamics is defined by an action of the general form

$$S_D[\phi] = \frac{1}{2} \int dx_i \, dy_i \; \phi(x_i) \, \mathcal{K}\big(x_i y_i^{-1}\big) \, \phi(y_i) + \frac{\lambda}{D+1} \int \prod_{i \neq j = 1}^{D+1} dx_{ij} \; \mathcal{V}\big(x_{ij} x_{ji}^{-1}\big) \times \phi(x_{1j}) \cdots \phi(x_{D+1j}),$$

where dx is an invariant measure on G, we use the notation $\phi(x_{1j}) = \phi(x_{12}, \ldots, x_{1D+1})$. $\mathcal{K}(X_i)$ is the kinetic and $\mathcal{V}(X_{ij})$ ($X_{ij} = x_{ij} x_{ji}^{-1}$) the interaction kernel, λ a coupling constant, the interaction is chosen to be homogeneous of degree $D + 1$. $\mathcal{K}$, $\mathcal{V}$ satisfy the invariance properties

$$\mathcal{K}(g X_i g') = \mathcal{K}(X_i), \quad \mathcal{V}\big(g_i X_{ij} g_j^{-1}\big) = \mathcal{V}(X_{ij}). \tag{4}$$

This implies that the action is invariant under the gauge transformations $\delta\phi(x_i) = \psi(x_i)$, where ψ is *any* function satisfying

$$\int_G dg \psi(g x_1, \ldots, g x_D) = 0. \tag{5}$$

This symmetry is gauge fixed if one restrict the field ϕ to satisfy $\phi(g x_i) = \phi(x_i)$. The action is also invariant under

$$\phi(x_1, \ldots, x_D) \to \phi(x_1 g, \ldots, x_D g). \tag{6}$$

The main interest of these theories resides in the following crucial properties they satisfy. Most of them are well established, some are new (property 4) and some (property 6) still conjectural. Altogether they give a picture of the relevance of GFT for background independent approach to quantum gravity and lead to the conclusion (or conjecture) that GFT provides a third quantization of gravity.

GFT Properties:

1. The Feynman graph of a D-GFT are cellular complexes dual to a D-dimensional triangulated topological spacetime.
2. The Feynman graph evaluation of a GFT are local spin foam models. Any local spin foam model can be obtained from a GFT.
3. Spin networks label polynomial gauge invariant operators of the GFT.
4. The tree-level two-point function of GFT gauge invariant operators gives a proposal for the physical scalar product. This proposal involves spacetime of trivial topology and is triangulation independent.

[10] It is also possible to generalize the definition to quantum groups or fuzzy group, we will restrict the discussion to compact group to avoid unnecessary technical subtleties.

5. The full two-point function of GFT gauge invariant operators gives a prescription for the quantum gravity amplitude including a sum over all topologies.
6. The possible loop divergences of GFT Feynman graphs are interpreted to be a consequence of a residual action of spacetime diffeomorphism on spin foam. One expect a relation between the renormalization group of GFT and the group of spacetime diffeomorphism.

2.2. GFT: Examples and Properties

In this section we give some examples and illustrate the properties listed above.

Some examples: The simplest examples comes from the choice

$$\mathcal{K}(x_i, y_i) = \int_G dg \prod_i \delta\big(x_i y_i^{-1} g\big), \quad \mathcal{V}(X_{ij}) = \int \prod_i dg_i \prod_{i<j} \delta\big(g_i X_{ij} g_j^{-1}\big), \tag{7}$$

$\delta(\cdot)$ is the delta function on G and the integrals insure the gauge invariance (5).

If one further restrict to dimension $D = 2$ the symmetry property implies that $\phi(g_1, g_2) = \tilde{\phi}(x_1^{-1} x_2)$. $\tilde{\phi}$ being a function on the group can be expanded in Fourier modes. Let's consider $G = \mathrm{SU}(2)$, denote by V_j the spin j representation, d_j its dimension, $D^j(x) \in \mathrm{End(V_j)}$ the group matrix element and define

$$\Phi_j \equiv \int dx\, \tilde{\phi}(x) D^j(x^{-1}), \quad \tilde{\phi}(x) = \sum_j d_j \mathrm{Tr}(\Phi_j D^j(x)). \tag{8}$$

One can readily see Livine *et al.* (2001) that the GFT reduces to a sum of matrix models:

$$S_2[\phi] = \sum_j d_j \left(\mathrm{Tr}\big(\Phi_j^2\big) + \frac{\lambda}{3} \mathrm{Tr}\big(\Phi_j^3\big) \right). \tag{9}$$

It is well known that the Feynman graph expansion of a matrix model is expressed in terms of fat graphs (Di Francesco *et al.*, 1995), each edge can be represented as a double line each one carrying a matrix index, the trivalent interaction implies that this graph is dual to a triangulation of a two-dimensional closed surface. Moreover if one compute the Feynman evaluation of a genus g diagram Γ diagram we find

$$I(\Gamma) = \sum_j d_j^{2-2g}, \tag{10}$$

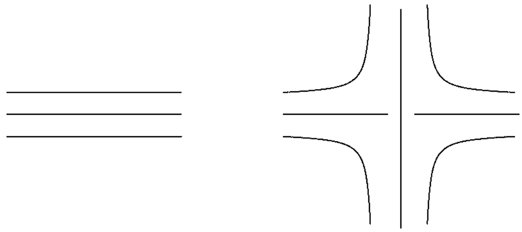

Fig. 1. Graphical representation of the propagator and interaction of a 3-GFT.

which is the evaluation of the partition function of topological BF theory in two dimension on a surface of genus g.[11] This result exemplifies the properties (1, 2). This property generalizes to any D, the Feynman graph evaluation of the example (7) gives the partition function of BF theory in dimension D.

Property 1. To illustrate this property in higher dimension let's consider the case of dimension $D = 3$, the field ϕ possess three arguments, so each edge of a Feynman graph possesses three strands running parallel to it, four edges meet at each vertex and the form of the interaction $\mathcal{V}$ forces the strands to recombine as in Fig. 1.

Each strand of the graph form a closed loop which can be interpreted as the boundary of a two-dimensional disk. These data are enough to reconstruct a topological two-dimensional complex $\mathcal{F}$, the vertices and edges of this complex correspond to vertices and edges or the Feynman graph, the boundary of the faces of $\mathcal{F}$ correspond to the strands of the Feynman graph. As we have already emphasized we can reconstruct a triangulated three-dimensional pseudo-manifold from such data.

It can be understood as follows: The three strands running along the edges can be understood to be dual to a triangle and the propagator gives a prescription for the gluing of two triangles. At the vertex four triangles meet and their gluing form a tetrahedra (see Fig. 2). With this interpretation the Feynman graph of a GFT is clearly dual to a three-dimensional triangulation. This is true in any dimension (De Pietri, 2001; De Pietri and Petronio, 2000; De Pietri *et al.*, 2000).

This means that the perturbative expansion of the partition function can be expressed as a sum over two complexes

$$\mathcal{Z} = \int \mathcal{D}\phi e^{-S_D[\Phi]} = \sum_{\mathcal{F}} \frac{\lambda^{|V|}}{\text{sym}(\mathcal{F})} I(\mathcal{F}), \qquad (11)$$

[11] If one choose the propagator $\mathcal{K}(x_i, y_i) = \int_G dg \prod_i \delta_\epsilon(x_i y_i^{-1} g)$ with δ_ϵ the heat kernel on the group, $(\partial_\epsilon + \Delta_G)\delta_\epsilon = 0$, $\delta_0(g) = \delta(g)$ we obtain the partition function of 2D Yang–Mills theory as a Feynman graph evaluation (Witten, 1991).

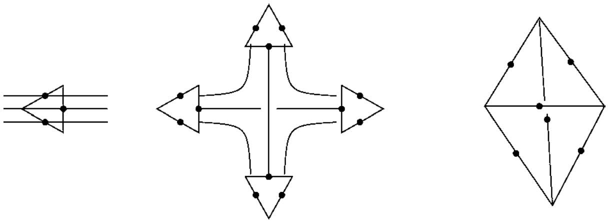

Fig. 2. Triangulation generated by Feynman diagrams.

where the sum is over two complex, $|V|$ is the number of vertices of $\mathcal{F}$, $\mathrm{sym}(\mathcal{F})$ the symmetry factor of $\mathcal{F}$ and $I(\mathcal{F})$ the GFT Feynman graph evaluation of the complex $\mathcal{F}$.

Property 2. Since the field ϕ is a function on D copies of the group it can be expanded in Fourier modes (8), the "momentum" of this field are spins which circulate along the strands of the Feynman graph or equivalently which label the faces of the complex $\mathcal{F}$. From the quantum gravity point-of-view this spin is interpreted as a quanta of area carried by the face. The computation of the Feynman graph in "momentum" space contains contributions from the Fourier modes of the propagator which gives edges amplitude A_{e}; from the Fourier mode of the interaction kernel which gives a vertex amplitude A_{v} and from the trace over the representation circulating on each face. Overall, the evaluation $I(\mathcal{F})$ can be expressed as a local spin foam model (2).

To precise this correspondence let's give a geometrical interpretation of the invariance property (4) of the interaction kernel $\mathcal{V}$: Let Γ_D be the graph of a D-dimensional simplex which consists of $D+1$ vertices and $D(D+1)/2$ edges. $\mathcal{V}$ is a function of group elements associated with the edges of Γ_D which is invariant under an action of the gauge group at the vertices of Γ_D. We denote the space of such function by $L^2(\Gamma_D)$. $L^2(\Gamma_D)$ admits an orthonormal basis labeled by spin networks $(\Gamma_D, j_{ij}, \iota_i)$, where j_{ij} are spins labeling the edges and ι_i are intertwiners labeling the vertices of Γ_D. Given $(\Gamma_D, j_{ij}, \iota_i)$, we can uniquely construct a spin network functional $\Theta_{j_{ij},\iota_i}(X_{ij})$.[12] The interaction kernel can be expanded in terms of this basis as follows

$$A_{\mathrm{v}}(j_{ij}, \iota_i) = \int \prod_{i<j} dX_{ij} \mathcal{V}(X_{ij}) \Theta_{j_{ij},\iota_i}(X_{ij}),$$

$$\mathcal{V}(X_{ij}) = \sum_{j_{ij},\iota_i} \prod_{i<j} d_{j_{ij}} A_{\mathrm{v}}(j_{ij}, \iota_i) \Theta_{j_{ij},\iota_i}(X_{ij}). \tag{12}$$

[12] Given a spin network $(\Gamma, j_{\mathrm{e}}, \iota_{\mathrm{v}})$ the spin network functional $\Theta_{\Gamma,j_{\mathrm{e}},\iota_{\mathrm{v}}}(x_{\mathrm{e}})$ is obtained by contracting the matrix elements of x_{e} in the representation j_{e} with the intertwiners ι_{v} according to the topology of the graph Γ. By construction this functional is invariant under the action of the group at each vertex of Γ.

Similarly, the quadratic kernel $\mathcal{K}(X_i)$ can be expanded in terms of the spin network functional $\Phi_{J_i}(X_i)$ associated with the "theta" graph which consists of two vertex joined by D edges:

$$1/A_{\mathrm{e}}(J_i) = \int \prod_i dX_i \mathcal{K}(X_i)\Theta_{J_i}(X_i) \quad \mathcal{K}(X_{ij}) = \sum_{J_i}\prod_i d_{J_i} 1/A_{\mathrm{e}}(J_i)\Theta_{J_i}(X_i). \tag{13}$$

It is now a direct computation to show that $I(\mathcal{F})$ is expressed as a local spin foam model with the edges and vertex amplitude determined by (12, 13) and the face amplitude being the dimension of the representation labeling the face.[13] Conversely, given a spin foam model we can reconstruct a GFT via (12, 13). This establishes the equivalence or duality between spin foam models and GFT, which was first proven in Reisenberger and Rovelli (2001). One can now check that the example (7) gives in three dimension the Ponzano–Regge model and in higher dimensions the discretization of topological BF. With this prescription we can also reconstruct from the Barrett–Crane amplitude (3) the interaction kernel:

$$\mathcal{V}_{BC}(X_{ij}) = \int_G \prod_i dg_i dh_i \int_H \prod_{i\neq j} du_{ij} \prod_{i<j} \delta(g_i u_{ij} h_i X_{ij} (g_j u_{ji} h_j)^{-1}), \tag{14}$$

where $G = \mathrm{SO}(4)$ and $H = \mathrm{SO}(3) \subset \mathrm{SO}(4)$.

Property 3. We have discussed so far only the partition function of the GFT, but one should also consider expectation value of GFT operators. The physical operators $\mathcal{O}(\phi)$ should be gauge invariant under (5, 6). Such operators can be constructed with the help of spin network: Let's consider a spin network $(\Gamma, j_{\mathrm{e}}, \iota_{\mathrm{v}})$ such that all its vertices have valency D and let's denote $\Theta_{(\Gamma, j_{\mathrm{e}}, \iota_{\mathrm{v}})}(x_{\mathrm{e}})$ the corresponding spin network functional (see footnote 12). We denote by V_Γ, E_Γ the set of vertices of Γ and define the observable of the D-GFT

$$\mathcal{O}_{(\Gamma, j_{\mathrm{e}}, \iota_{\mathrm{v}})}(\phi) = \int_G \prod_{(ij)\in E_\Gamma} dx_{ij}\, dx_{ji}\, \Theta_{(\Gamma, j_{\mathrm{e}}, \iota_{\mathrm{v}})}(x_{ij}(x_{ji})^{-1}) \prod_{i\in V_\Gamma} \phi(x_{ij}). \tag{15}$$

The element x_{ij} are associated to all the edges of Γ meeting at the vertex i, by construction there is always D such elements. This observable is homogenous in ϕ, the degree of homogeneity being the number of vertices of Γ. It is straightforward to check that this observable respect the symmetries of the GFT.

Property 4. We can now come back to the original problem, that is the construction of the physical scalar product. Since spin networks label operators of the GFT, we propose to define this scalar product as the evaluation of the GFT two point function in the *tree level* truncation. Namely, given two D valent spin

[13] It is possible to produce different face amplitude by modifying the symmetry properties of the action (De Pietri *et al.*, 2000).

network Γ_1, Γ_2 having N_1, N_2 vertices we define

$$\langle \Gamma_1 | \Gamma_2 \rangle_0 \equiv \langle \mathcal{O}_{\Gamma_1} | \mathcal{O}_{\Gamma_2} \rangle_{Tree} = \sum_{\mathcal{F} \in \mathcal{T}_{N_1,N_2}} \frac{I(\mathcal{F})}{\mathrm{sym}(\mathcal{F})}. \tag{16}$$

$\mathcal{T}_{N_1,N_2}$ denote the space of GFT Feynman graphs supported on connected trees having N_1 initial univalent vertices and N_2 final univalent vertices, we sum over all of them.

This simple proposal for the scalar product does not depend on a particular triangulation, and therefore it addresses one of the main shortcomings of the spin foam approach. This product satisfies two crucial properties: First, it is well defined and finite, since it is a tree-level evaluation no infinite summation are involved. Second, it is positive but not strictly positive, it possesses a kernel. This kernel should be expected, since the physical scalar product of quantum gravity (1) computes the matrix elements of the projector on the kernel of the hamiltonian constraints. This means that any vector in the image of the hamiltonian constraint belongs to the kernel of (1). In our case, one can show that the GFT scalar product (1) has a kernel which is in the image of the GFT equation of motion. Namely, the following gauge invariant observable

$$\delta \mathcal{O}_{(\Gamma, j_e, \iota_v)}(\phi) = \int_G \prod_i dx_i \left(\mathcal{K}^{-1} \frac{\delta S[\phi]}{\delta \phi} \right)(x_i) \frac{\delta \mathcal{O}_{(\Gamma, j_e, \iota_v)}(\phi)}{\delta \phi(x_i)}, \tag{17}$$

is in the kernel of (16). In this formula $\mathcal{K}^{-1}$ is the propagator, it is convoluted with the equation of motion. We can expand this observable as a linear combination of spin network observables, the first term in the expansion of $\delta \mathcal{O}_\Gamma$ is $\mathcal{O}_\Gamma$ the other terms are spin network observables containing D more fields.

The physical Hilbert space can be constructed as an application of the Gelfand–Naimark–Segel theorem. It is obtained from the kinematical Hilbert space spanned by spin networks, by quotienting out the vectors in the kernel of (16): $\mathcal{H}_{\mathrm{phys}} = \mathcal{H}/\mathrm{Ker}\langle \cdot | \cdot \rangle_0$ (Perez and Rovelli, 2001). The induced scalar product on $\mathcal{H}_{\mathrm{phys}}$ is positive definite.

The product (16) involves only tree Feynman graphs. Using the correspondence between GFT Feynman graphs and discrete manifolds one sees that all the manifolds involved in the sum are of the same topology and describe a ball on the boundary of which the operators are inserted. Since this product (16) is independent of the choice of triangulation it can be thought as a "continuous" scalar product. One might worry that this is realized without taking any sort of continuum or refinement limit and therefore that this prescription describes some sort of topological field theory. This is not the case, in this prescription the complexity of the spacetime triangulation involved in the summation grows with the complexity of the boundary spin network state. If one consider highly complicated spin network states that approach a continuum geometry, the corresponding spin foams (discrete spacetimes) involved in the summation are also highly complicated and good approximation of a continuous geometry. Also, we have seen that $\delta\mathcal{O}_\Gamma$ is a linear combination of $\mathcal{O}_\Gamma$ and higher order spin network operators. We can therefore replace in the computation of the scalar product, the state Γ by a linear combination of spin network states of higher degree, and continue this replacement in each new terms ad infinitum, therefore ending with a expression for the scalar product in terms of a linear combination of arbitrarily fine triangulations which gives a "true" continuum limit expression of the scalar product. This use of the kernel express the fact that a subset of the Hilbert space based on a fine triangulation can be effectively described in terms of states leaving on a coarser one. The theory is not topological if this subset is a proper subspace (this is the case for the Barrett–Crane model for instance).

This proposal for a physical scalar product in the context of loop quantum gravity closes in some sense a long quest starting from the construction of the hamiltonian constraints, a search for its solutions and for the physical scalar product. It gives us a hint on the answer to the last question (which contains the others), it doesn"t end the quest but provides a new starting point. The problem is now to understand the dynamical content which is contained in such a proposal and to see wether at least one GFT (the Barrett–Crane one for instance) possesses the right dynamical content and can reproduce the physics of general relativity in the infrared. This question is not new, but with the help of the GFT it can be asked for the first time in terms of a proposed physical scalar product. The difficulty resides in the fact that the dynamics is encoded in terms of spin networks transition amplitudes, a language far remote from semiclassical physics and one need to design criteria to select the right model or to test and eventually refute a proposed one.

Property 5. The previous scalar product can be naturally extended to include a sum over all Feynman graphs of the GFT, this was the original proposal (De Pietri *et al.*, 2000), the gravity amplitude is in this case

$$\begin{aligned}\langle\Gamma_1|\Gamma_2\rangle_\lambda &= \ln\left[\frac{\lambda^{-\frac{N_1+N_2}{D-1}}}{\mathcal{Z}}\int \mathcal{D}\phi\, \mathcal{O}_{\Gamma_1}(\phi)\mathcal{O}_{\Gamma_2}(\phi)e^{-S_D[\Phi]}\right]\\ &= \sum_{\mathcal{F},\partial\mathcal{F}=\Gamma_1\cup\Gamma_2}\frac{\lambda^{|V|-\frac{N_1+N_2}{D-1}}}{\mathrm{sym}(\mathcal{F})}I(\mathcal{F}).\end{aligned} \tag{18}$$

where $|V|$ is the number of vertices of $\mathcal{F}$, this correspond to the number of D simplex of the dual triangulation, the sum is over connected graph matching the given spin network on the boundary, N_1, N_2 are the degree of homogeneity of the operators $\mathcal{O}_{\Gamma_1}$, $\mathcal{O}_{\Gamma_2}$ and $\mathcal{Z}$ is the partition function (11). The coupling parameter λ weights, in the perturbation expansion, the size of the discrete spacetime. It can be given another interpretation: Let's define $\alpha = \lambda^{1/D-1}$ and let's redefine the fields $\tilde{\phi} = \alpha\phi$, the action becomes $S_\lambda[\phi] = 1/\alpha^2 \tilde{S}[\tilde{\phi}]$ where $\tilde{S} = S_{\lambda=1}$ is independent of the coupling constant. The amplitude can be expanded in α, $\langle\Gamma_1|\Gamma_2\rangle_\alpha = \alpha^{-2}\sum_i \alpha^{2i}\langle\Gamma_1|\Gamma_2\rangle_i$ where $\langle\Gamma_1|\Gamma_2\rangle_i$ is a sum of GFT Feynman graphs containing i loops. From the space time point-of-view adding a loop to a GFT Feynman diagram amounts to adding a handle to the discrete manifold. Hence α controls the strength of topology change. In the limit $\alpha = 0$, we recover the classical evaluation (16) where topology change is suppressed. This can be also understood by looking at the Schwinger–Dyson equation of motion. Let's focus for simplicity on the nucleation amplitude where $\Gamma_1 = \emptyset$, $\mathcal{O}_{\Gamma_1} = 1$ which described the creation of a spacetime from nothing. The Schwinger–Dyson equation reads

$$\langle\delta\mathcal{O}_\Gamma\rangle_\alpha = \alpha^2\langle\delta^2\mathcal{O}_\Gamma\rangle_\alpha, \tag{19}$$

where $\delta\mathcal{O}_\Gamma$ (17) is in the kernel of the physical scalar product and

$$\delta^2\mathcal{O}_\Gamma \equiv \int_G \prod_i dx_i dy_j \mathcal{K}^{-1}(x_i, y_j)\frac{\delta^2\mathcal{O}_\Gamma(\phi)}{\delta\phi(x_i)\delta\phi(y_j)}. \tag{20}$$

We have seen that the field ϕ is dual to a $D-1$ simplex, the operator $\delta\mathcal{O}_\Gamma$ corresponds to a sum of spin network boundary states, one being a triangulation dual to Γ the others obtained by subdividing one of the $D-1$ simplex of this triangulation. The operator $\delta^2\mathcal{O}_\Gamma$ deletes two $D-1$ simplices and glues the resulting holes together thus creating an handle. This handle creation is weighted by α^2.

Note that the scalar product $\langle\cdot|\cdot\rangle_\alpha$ is strictly positive, the states $\delta\mathcal{O}_\Gamma$ which are in the kernel of $\langle\cdot|\cdot\rangle_0$ now generates topology change (19).

Since we now include Feynman graphs with loops we have to worry about potential perturbative divergences due to the Feynman graph evaluation. A careful analysis shows that the potential divergences of the GFT are not associated with loops but with higher-dimensional analogs: The so-called "bubbles" of the spin foam (Perez and Rovelli, 2001). A bubble is a collection of faces of the two complex $\mathcal{F}$ which forms a closed surface. Each time a bubble appears the sum over spins (GFT momenta) is unrestricted and potentially infinite. A remarkable finiteness result was proven in Perez (2001) for the Barrett–Crane model. It was shown that if one take the interaction kernel (14) and the propagator (7), there are no divergences arising in the computation of Feynman graph (associated with regular two-dimensional complex), the corresponding GFT is super-renormalizable in this case.

There is also a possibility of potential non-perturbative divergences which arises from the sum over topology. It is well known that the number of triangulated manifold of arbitrary topology grows factorially with the number of building block and the sum (18) is therefore not convergent for any nonzero value of α. This is not surprising from the GFT point of view since we know that a perturbative expansion should be interpreted as an asymptotic series, not a convergent series.

In some cases, this series is uniquely Borel summable and the GFT provides a nonperturbative definition for the sum over all topologies. This was shown to be true in the context of three-dimensional gravity where Borel summability of a mild modification of the Boulatov model was proven. It is not known wether this can be achieved in four-dimensional gravity.

Property 6. As we already mentioned, the key open problem is to gain an understanding of the low-energy effective physics from the GFT. One proposal for addressing this question is to focus on the issue of the diffeomorphism symmetry. We know that any theory which depends on a metric reproduces gravity in the infrared if it is invariant under spacetime diffeomorphism. In loop quantum gravity, spin networks label the gravitational degree of freedom, this suggests that any spin foam model which can be shown to respect spacetime diffeomorphism will contain gravity in a low energy limit. The problem is therefore to have a proper understanding of the action of spacetime diffeomorphism on spin foam models. It was argued in Freidel and Louapre (2003) (and exemplified in the context of three-dimensional gravity) that diffeomorphism symmetry should act as a gauge symmetry on the spin foam amplitudes. This means that the initial spin foam amplitudes 2 which are not gauge fixed with respect to this symmetry and which do not break diffeomorphism symmetry should possess divergences coming from the ungauged integration over the diffeomorphism gauge group. Diffeomorphism symmetry is due to the Bianchi identity which is a three form on spacetime and then couples to the bubbles of spin foams. Therefore diffeomorphism symmetry should manifest itself in the bubble divergences.

This analysis leads to the conclusion that the LBarret–Crane GFT model proposed in Perez and Rovelli (2001); Oriti and Williams (2001) which has no bubble divergences is not a satisfactory model.[14] From the GFT point-of-view the bubble divergences is analogous to the loop divergences in usual field theory. We know that such divergences are the manifestation of a nontrivial renormalization group acting on the parameter space of field theory.

Since, as we have seen in this note, any relevant property of spin foam model admits its dual formulation in the GFT, this strongly suggest that a proper understanding of the action of the diffeomorphism group on spin foam models is related to a proper understanding of the GFT renormalization group.

[14] The prescription of Perez and Rovelli (2001); Oriti and Williams (2001) differs from the original prescription (De Pietri *et al.*, 2000) (which possess bubble divergences) by the choice of the kinetic term of the GFT. This kinetic term controls the way different vertex amplitude are glued together. There is a large consensus and good understanding of the Barrett–Crane vertex amplitude but so far, no general agreement on the choice of the kinetic term has been reached. Different choices leads to different properties with respect to the bubble divergences. Of course this argumentation is not yet conclusive since it contain hypothesis and unresolved issues.

REFERENCES

Ambjorn, J., Durhuus, B., and Jonsson, T. (1991). Three-dimensional simplicial quantum gravity and generalized matrix models. *Modern Physics Letters A* **6**, 1133.

Ashtekar, A. and Lewandowski, J. (2004). Background independent quantum gravity: A status report. *Classical Quantum Gravity* **21**, R53 (2004) [arXiv:gr-qc/0404018].

Baez, J. C. (1996). Four-Dimensional BF theory with cosmological term as a topological quantum field theory. *Letters of Mathematical Physics* **38**, 129 [arXiv:q-alg/9507006].

Baez, J. C. (1998). Spin foam models. *Classical Quantum Gravity* **15**, 1827 [arXiv:gr-qc/9709052].

Barrett, J. W. (1998). The Classical evaluation of relativistic spin networks. *Advances in Theoretical Mathematical Physics* **2**, 593 [arXiv:math.qa/9803063].

Barrett, J. W. and Crane, L. (1998). Relativistic spin networks and quantum gravity. *Journal of Mathematical Physics* **39**, 3296 [arXiv:gr-qc/9709028].

Barrett, J. W. and Crane, L. (2000). A Lorentzian signature model for quantum general relativity. *Classical Quantum Gravity* **17**, 3101 [arXiv:gr-qc/9904025].

Boulatov, D. V. (1992). A Model of three-dimensional lattice gravity. *Modern Physics Letters A* **7**, 1629 [arXiv:hep-th/9202074].

De Pietri, R. and Petronio, C. (2000). Feynman diagrams of generalized matrix models and the associated manifolds in dimension 4. *Journal of Mathematical Physics* **41**, 6671 [arXiv:gr-qc/0004045].

De Pietri, R. (2001). Matrix model formulation of four dimensional gravity. *Nuclear Physics Proceedings Supplementary* **94**, 697. [arXiv:hep-lat/0011033].

De Pietri, R. and Freidel, L. (1999). so(4) Plebanski Action and Relativistic Spin Foam Model. *Classical Quantum Gravity* **16**, 2187 [arXiv:gr-qc/9804071].

De Pietri, R., Freidel, L., Krasnov, K., and Rovelli, C. (2000). Barrett–Crane model from a Boulatov-Ooguri field theory over a homogeneous space. *Nuclear Physics B* **574**, 785 [arXiv: hep-th/9907154].

Di Francesco, P., Ginsparg, P. H., and Zinn-Justin, J. (1995). 2-D Gravity and random matrices. *Physical Report* **254**, 1 [arXiv:hep-th/9306153].

Freidel, L. (2000). A Ponzano–Regge model of Lorentzian 3-dimensional gravity. *Nuclear Physics Proceedings Supplementary* **88**, 237 [arXiv:gr-qc/0102098].

Freidel, L. and Krasnov, K. (1999). Spin foam models and the classical action principle. *Advances in Theoretical Mathematical Physics* **2**, 1183 [arXiv:hep-th/9807092].

Freidel, L. and Krasnov, K. (2000). Simple spin networks as Feynman graphs. *Journal of Mathematical Physics* **41**, 1681 [arXiv:hep-th/9903192].

Freidel, L. and Louapre, D. (2003). Diffeomorphisms and spin foam models. *Nuclear Physics B* **662**, 279 [arXiv:gr-qc/0212001].

Freidel, L., Krasnov, K., and Puzio, R. (1999). BF description of higher-dimensional gravity theories. *Advances in Theoretical Mathematical Physics* **3**, 1289 [arXiv:hep-th/9901069].

Livine, R., Perez, A., and Rovelli, C. (2001) 2d manifold-independent spinfoam theory, arXiv: gr-qc/0102051.

Markopoulou, F. and Smolin, L. (1997). Causal evolution of spin networks. *Nuclear Physics B* **508**, 409 [arXiv:gr-qc/9702025].

Mikovic, A. (2003). Spin foam models of Yang–Mills theory coupled to gravity. *Classical Quantum Gravity* **20**, 239 [arXiv:gr-qc/0210051].

Okolow, A. and Lewandowski, J. (2003). Diffeomorphism covariant representations of the holonomy-flux *-algebra. *Classical Quantum Gravity* **20**, 3543 [arXiv:gr-qc/0302059].

Ooguri, H. (1992). Topological lattice models in four-dimensions. *Modern Physics Letters A* **7**, 2799 [arXiv:hep-th/9205090].

Oriti, D. and Pfeiffer, H. (2002). A spin foam model for pure gauge theory coupled to quantum gravity. *Physical Review D* **66**, 124010 [arXiv:gr-qc/0207041].

Oriti, D. and Williams, R. M. (2001). Gluing 4-simplices: A derivation of the Barrett–Crane spin foam model for Euclidean quantum gravity. *Physical Review D* **63**, 024022 [arXiv:gr-qc/0010031].

Perez, A. (2001). Finiteness of a spinfoam model for Euclidean quantum general relativity. *Nuclear Physics B* **599**, 427 [arXiv:gr-qc/0011058].

Perez, A. (2003). Spin foam models for quantum gravity, Class. *Quantum Gravity* **20**, R43 [arXiv: gr-qc/0301113].

Perez, A. and Rovelli, C. (2001). Observables in quantum gravity, arXiv:gr-qc/0104034.

Perez, A. and Rovelli, C. (2001). A spin foam model without bubble divergences. *Nuclear Physics B* **599**, 255 [arXiv:gr-qc/0006107].

Ponzano, G. and Regge, T. (1968). Semiclassical limit of Racah coefficients. In *Spectroscopic and Group Theoretical Methods in Physics, Racah Memorial Volume*, F. Block *et al*. eds., North Holland, Amsterdam.

Reisenberger, M. P. (1994). World sheet formulations of gauge theories and gravity, arXiv:gr-qc/9412035.

Reisenberger, M. P. and Rovelli, C. (1997). *Sum over surfaces* form of loop quantum gravity. *Physical Review D: Particles and Fields* **56**, 3490 [arXiv:gr-qc/9612035].

Reisenberger, M. P. and Rovelli, C. (2001). Spacetime as a Feynman diagram: The connection formulation. *Classical and Quantum Gravity* **18**, 121 [arXiv:gr-qc/0002095].

Sahlmann, H. (2002). Some comments on the representation theory of the algebra underlying loop quantum gravity. arXiv:gr-qc/0207111.

Sahlmann, H. and Thiemann, T. (2003). Irreducibility of the Ashtekar–Isham–Lewandowski representation. arXiv:gr-qc/0303074.

Thiemann, T. (1998). Quantum spin dynamics (QSD). *Classical and Quantum Gravity* **15**, 839 [arXiv:gr-qc/9606089].

Turaev, V. G. and Viro, O. Y. (1992). State sum invariants of 3 manifolds and quantum 6j symbols. *Topology* **31**, 865.

Witten, E. (1988). $(2 + 1)$-Dimensional Gravity as an exactly soluble system. *Nuclear Physics B* **311**, 46.

Witten, E. (1991). On quantum gauge theories in two-dimensions. *Communications in Mathematical Physics* **141**, 153.